Mathematics 222B Lecture 2 Notes

Daniel Raban

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1 A Priori Estimates and Approximation Theorems

1.1 Relationship between a priori estimates, existence, and uniqueness

Last time, we were investigating the question "Why study Sobolev spaces as Banach spaces?" We made a digression into functional analysis:

If X and Y are Banach spaces and $P : X \to Y$ is bounded and linear, we had 2 concerns:

- (Existence) Given $f \in T$, does there exists a $u \in X$ such that $Pu = f$?
- (Uniqueness) Given $u \in X$ such that $Pu = 0$, does $u = 0$?

These two problems are related to each other by duality.

Remark 1.1. Here is a concrete thing to keep in mind: Often, we prove a priori estimates for a PDE, i.e. if $u \in X$ with $Pu = f$, then $\|u\|_X \leq C\|f\|_Y$.

Proposition 1.1. *Let X, Y be Banach spaces, and let $P : X \to Y$ be a bounded, linear operator. Denote by $P^* : Y^* \to X^*$ the adjoint of P, i.e. $\langle v, Pu \rangle = \langle P^*v, u \rangle$ for all $u \in X, v \in Y^*$. Suppose there exists a constant $C > 0$ such that $\|u\|_X \leq C\|Pu\|_Y$ for all $u \in X$. Then*

1. *(Uniqueness for $Pu = f$) If $u \in X$ and $Pu = 0$, then $u = 0$.*
2. *(Existence for $P^*v = g$) For all $g \in X^*$, there exists a $v \in Y^*$ such that $P^*v = g$ and $\|v\|_{Y^*} \leq C\|g\|_{X^*}$.*

Proof. Here is the proof of 2, via the Hahn-Banach theorem. We want to find $v \in Y^*$ such that $P^*v = g$, which is equivalent to $\langle P^*v, u \rangle = \langle g, u \rangle$ for all $u \in X$. The left side is $\langle v, Pu \rangle$, so we will start with a subspace of elements of the form Pu.

Define $\ell : P(X) \to \mathbb{R}$ by the relation

$$\ell(Pu) = \langle g, u \rangle.$$

Note that since P is injective, this ℓ is well-defined. This is bounded because or $\|Pu\|_Y \leq 1$,

$$\begin{aligned} \ell(Pu)| = |\langle g, u\rangle| &\leq \|g\|_{X^*}\|u\|_X \\ &\leq C\|g\|_{X^*}\|Pu\|_Y \\ &\leq C\|g\|_{X^*} \end{aligned}$$

So Hahn-Banach says that there is a $v \in Y^*$ such that

$$\langle v, Pu\rangle = \ell(pu) = \langle g, u\rangle \qquad \forall u \in X$$

and $\|v\|_{Y^*} \leq C\|g\|_{X^*}$. ◻

What about existence for the original problem $Pu = f$? Let us take an easy way out and assume that X is **reflexive** ($X \to (X^*)^*$ sending $u \mapsto (u \mapsto \langle v, u\rangle)$ is an isomorphism).

Proposition 1.2. *Let X, Y be Banach spaces, and let $P : X \to Y$ be a bounded, linear operator. Suppose $\|v_{Y^*}\| \leq C\|P^*v\|_{X^*}$. Then*

1. *(Uniqueness for $P^*v = g$) If $v \in Y^*$ and $P^*v = 0$, then $v = 0$.*
2. *(Existence for $Pu = f$) For all $f \in Y$, there exists a $u \in X$ such that $Pu = f$ and $\|u\|_X \leq C\|f\|_Y$.*

Proof. Same as before. Construct $u \in X$ by constructing a bounded linear functional on X^* (because $X = (X^*)^*$ by reflexivity. ◻

Remark 1.2. All Sobolev spaces $W_0^{k,p}(U)$ with $1 < p < \infty$ are reflexive.

Remark 1.3.

$$(\operatorname{ran} P)^\perp = \ker P^*, \qquad \ker P =^\perp (\operatorname{ran} P^*).$$

Here, we mean *annihilators.*

Definition 1.1. Given $U \subseteq Y$, the **annihilator** of U is $U^\perp = \{v \in Y^* : \langle v, f\rangle = 0 \forall f \in U\}$. Given $V \subseteq X^*$, the **annihilator** of U is $U^\perp = \{u \in X : \langle g, u\rangle = 0\ \forall g \in V\}$.

As a consequence, if $\ker P^* = \{0\}$, then by Hahn-Banach,

$$(\operatorname{ran} P)^\perp = \{0\} \iff \overline{\operatorname{ran} P} = Y.$$

In the finite dimensional case, $\overline{\operatorname{ran} P} = \operatorname{ran} P$. Therefore, we get the well-known fact from linear algebra concerning the solvability of the problem $Ax = b$ with A a possibly non-square matrix:

$$(\text{for all } b\text{, there exists an } x \text{ such that } Ax = b) \iff (A^*y = 0 \implies y = 0),$$

$$(\text{for all } c\text{, there exists an } y \text{ such that } A^*y = c) \iff (Ax = 0 \implies x = 0).$$

However, in the infinite dimensional case, $\overline{\operatorname{ran} P} = \operatorname{ran} P$, so we can think of the annihilator as measuring how close these are.

Remark 1.4 (Qualitative vs quantitative)**.** There is no loss of generality in deriving existence for $Pu = f$ from the quantitative bound $\|v\|_{Y^*} \leq C\|P^*v\|_{X^*}$.

Proposition 1.3. *Let X, Y be Banach spaces and $P : X \to Y$ be a bounded linear operator. If $P(X) = T$, then there exists some $C > 0$ such that $\|v\|_{Y^*} \leq C\|P^*v\|_{X^*}$.*

Proof. By the open mapping theorem, $P(B_X)$, the image of the unit ball in X, is open and contains the origin. So there exists a $C > 0$ such that $P(B_X) \supseteq cB_Y$. Then

$$\begin{aligned}
\|P^*v\|_{X^*} &= \sup_{u:\|u\|_X \leq 1} |\langle P^*v, u\rangle| \\
&= \sup_{u \in \overline{B_X}} |\langle v, Pu\rangle| \\
&= \sup_{f \in P(\overline{B_X})} |\langle v, f\rangle| \\
&\geq \sup_{f \in cB_Y} |\langle v, f\rangle| \\
&\geq C\|v\|_{Y^*}.
\end{aligned}$$

□

Example 1.1. Let's try to solve the 1-dimensional Laplace equation

$$\begin{cases} -u'' = f & \text{in } (0,1) \\ u = 0 & \text{at } x = 0, 1. \end{cases}$$

We will investigate solvability in $H_0^1((0,1)) = \overline{C_c^\infty(0,1)}^{\|\cdot\|_{H^1}}$, where $\|u\|_{H^1}^2 = \|u_{L^2}^2 + \|u'\|_{L^2}^2$. Recall that $(H_0^1((0,1)))^* = H^{-1}(0,1)$. Then we have $Pu = -u''$ with domain $X = H_0^1((0,1))$ and codomain $Y = H^{-1}(0,1)$.

We claim that if $Pu = f$ for some $u \in X$ then $\|u\|_X \leq C\|f\|_Y$. This means that if $u \in H_0^1(0,1)$ satisfies the equation $-u'' = f$, then $\|u\|_{H^1} \leq C\|f\|_{H^{-1}}$.

Proof. To prove this bound, it suffices by density to consider $u \in C_c^\infty((0,1))$. Multiply both sides by u and integrate:

$$\int fu\, dx = \int -u''u\, dx$$

Since $u \in C_c^\infty((0,1))$ there are no boundary terms. So we may integrate by parts.

$$= \int (u')^2\, dx.$$

But how about $\|u\|_{L^2}$? Use the fact that u vanishes on the boundary:

$$u(x) = \int_0^x u'(x)\, dx.$$

Then for any $x \in (0,1)$, we can say

$$|u(x)| \leq \int_0^1 |u'(x')|\, dx' \overset{\text{Cauchy-Schwarz}}{\leq} \|u'\|_{L^2}^2.$$

We now have that

$$\begin{aligned} \|u\|_{H^1}^2 &\leq C|\langle f, u\rangle| \\ &\leq C\|f\|_{H^{-1}}\|u\|_{H^1}. \end{aligned}$$

Cancelling one factor of $\|u\|_{H^1}$ on each side gives $\|u\|_{H^1} \leq C\|f\|_{H^{-1}}$. $\square$

Combined with proposition 1 gives us that if $-u'' = 0$ and $u \in H_0^1((0,1))$, then $u = 0$. To use proposition 2, we need to compute P^*:

$$\langle P^* v, u\rangle = \langle v, Pu\rangle \qquad \forall v \in (H^{-1})^*, u \in H_0^1.$$

Note that by reflexivity of H_0^1, $(H^{-1})^* = H_0^1$. Let's write this out:

$$\langle v, Pu\rangle = \int_0^1 v(-u'')\, dx$$

To use integration by parts, do another density argument.

$$\begin{aligned} &= \int_0^1 v'u'\, dx \qquad (v \in H_0^1) \\ &= \int_0^1 -v''u\, dx \qquad (u \in H_0^1) \\ &= \langle P^* v, u\rangle. \end{aligned}$$

This tells us that $P^* v = -v''$ with domain $Y^* = H_0^1((0,1))$ and codomain $X^* = H_0^{-1}((0,1))$, so the problem is self-dual. So we get existence: for all $f \in H^{-1}$, there is a $u \in H_0^1$ such that $Pu = f$.

This is a pretty high-powered approach that works for a variety of problems. To prove quantitative estimates, we will in general use **Poincaré inequlities**.

1.2 Approximation by smooth functions and smooth partition of unity

There are two main tools we will use: convolution and mollifiers.

Lemma 1.1. *Let φ be smooth, compactly supported, and have $\int \varphi\, dx = 1$. Let $u \in L^p(\mathbb{R}^d)$ with $1 \leq p < \infty$. Denote **mollifiers** $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d}\varphi(x/\varepsilon)$ (so $\int \varphi_\varepsilon$). Then*

$$\|\varphi_\varepsilon u - u\|_{L^p} \xrightarrow{\varepsilon \to 0} 0,$$

where $\varphi_\varepsilon * u = \int \varphi_\varepsilon(x-y)u(y)\, dy$.

Proof. The key ingredient is the continuity of the translation operator on L^p. Define for $z \in \mathbb{R}^d$ and $u \in L^p$ the translation operator $\tau_z u(x) = u(x-z)$. Then

$$\lim_{|z|\to 0} \|\tau_z u - u\|_{L^p} = 0,$$

which you can check. Now

$$\varphi_\varepsilon * u(x) - u(x) = \int u(x-y)\varphi_\varepsilon(y)\,dy - u(x)$$

Since $\int \varphi_\varepsilon = 1$,

$$= \int (u(x-y) - u(x))\varphi_\varepsilon(y)\,dy.$$

Taking the L^p norm, we have

$$\begin{aligned} \|\varphi_\varepsilon * u(x) - u(x)\|_{L^p} &= \left\| \int (u(x-y) - u(x))\varphi_\varepsilon(y)\,dy \right\|_{L^p} \\ &\le \int \|u(\cdot - y) - u(\cdot)\|_{L^p} |\varphi_\varepsilon(y)|\,dy \end{aligned}$$

Since φ has compact support, $\operatorname{supp}\varphi_\varepsilon \to \{0\}$ as $\varepsilon \to \{0\}$. Thus, the integrand goes to 0 as $\varepsilon \to 0$. So we may apply the dominated convergence theorem to get

$$\xrightarrow{\varepsilon\to 0} 0.$$

□

This approximation is useful because $\varphi_\varepsilon * u$ is smooth.

Another useful tool is a smooth partition of unity:

Lemma 1.2. *Suppose $\{U_\alpha\}_{\alpha\in A}$ be an open covering of U in $\mathbb{R}^d$. There exists a **smooth partition of unity** $\{\chi_\alpha\}_{\alpha\in A}$ on U subordinate to $\{U_\alpha\}_{\alpha\in A}$, i.e.*

1. *$\sum_\alpha \chi_\alpha(x) = 1$ on U and for all $x \in U$ there exist only finitely many nonzero $\chi_\alpha(x)$*
2. *$\operatorname{supp}\chi_\alpha \subseteq U_\alpha$*
3. *χ_α is smooth.*

Proof. Start from a continuous partition of unity and apply the previous lemma to approximate by smooth functions. □